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THE INTERPOLATION OF LAGRANGE IN SPACE $\mathbb{R}^m$

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ABSTRACT

In this article, a formula is deduced in an original manner for interpolation in the real space of m dimensions ($\mathbb{R}^m$), inspired by the well-known Lagrange formula for real functions ($F(x)$) of a variable on the real line $\mathbb{R}$. The results that are shown here do not seem well known, at least not in Colombian university journals. These results have been searched by the author for some time, and he has not found them. In the end, he had to deduce them himself.

KEYWORDS: Numerical Analysis; Numerical Integration; Analytic Geometry in Space; Convex polyhedra.

LA INTERPOLACIÓN DE LAGRANGE EN EL ESPACIO $\mathbb{R}^m$

RESUMEN

En este artículo se deduce de manera original una fórmula de interpolación en el espacio real de m dimensiones ($\mathbb{R}^m$), inspirada en la conocida fórmula de Lagrange para funciones reales ($F(x)$) de una variable, es decir en la recta real $\mathbb{R}$. Los resultados que aquí se obtienen no parecen ser muy conocidos, al menos, en los medios universitarios de Colombia. El autor los ha buscado durante mucho tiempo, sin hallarlos. Finalmente tuvo que deducirlos por sí solo.

PALABRAS CLAVE: análisis numérico; integracion numérica; Geometría Analítica del Espacio; Cuerpos convexos.

A INTERPOLAÇÃO DE LAGRANGE NO ESPAÇO $\mathbb{R}^m$

RESUMO

Este artigo se deduz de uma forma original uma fórmula de interpolação no espaço real de m dimensões reais ($\mathbb{R}^m$), inspirada pela conhecida fórmula de Lagrange para funções reais ($F(x)$) de uma variável, ou seja, na reta real $\mathbb{R}$. Os resultados aqui obtidos não parecem ser bem conhecidos, pelo menos na academia da Colômbia. O autor os tem procurado por um longo tempo, sem encontrá-los. Finalmente teve que deduzi-los sozinho.

PALAVRAS-CHAVE: Análise Numérica; Integração Numérica; Geometria Analítica no Espaço; Poliedros convexos.

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1. INTRODUCTION

The Lagrange formula for interpolation for real functions of a single variable can be found in all books, elementary and advanced, on numerical analysis. In some of the more advanced books they use it to deduce other results that are very important for numerical analysis and very useful for applications in engineering, actuarial science, economics, and other real sciences. It can be seen, for example, in what are now classic books from Hammign, Ralston, Scheid, Willers, Mineur, Pearson, and others that are cited in the bibliography.

But, unfortunately, none of these now famous books say how this formula can be extended to continuous functions of two variables, that is in the form $F(x, y)$, defined in a continuous domain of plane OXY. They do not even suggest to readers the possibilities of doing so. Of course, even less space is spent on the continuous functions $F(x, y, z)$ of three variables in tridimensional space $OXYZ$ (or, in space $\mathbb{R}^3$, as modern books call it). The author has searched for several years, in books and in magazines, advanced and elementary, in different fields of algebra, analysis, and analytic geometry, to see if any of them present Lagrange's formula in two or more variables, or if they even suggest that it is possible to construct it. But he has found nothing. Therefore, he has proceeded to do this work and deduce the Lagrange formulas to interpolate the functions of two, three, and n dimensions ($n \geq 2$) that are presented in this article.

2. LAGRANGE'S FORMULA IN ONE DIMENSION

1. Let's remember what Lagrange's formula is in one dimension: we have points on the straight line N ($n \geq 2$) with abscissae $x_1, x_2, \dots, x_N$ which, if desired, can be considered as well-known real numbers in their numerical value, that is, they can be fed into a computer with precision and with the exact number of digits as appropriate. Furthermore, in order to facilitate nomenclature, let's suppose, without a loss of generality, that $x_1 < x_2 < x_3 < \cdots < x_N$, siendo, being, of course, x_1, the smallest of all, and x_N the largest of all. We also have N numerical values $u_1, u_2, \dots, u_N$ corresponding to a variable dependent on the points x_i (being $i = 1,2, \dots, N$) mentioned above. These last numerical values, if desired, can be considered as known data, without errors in measurement, which can also be fed into a computer with as many exact digits as required. In some situations, the values u_i can be those that correspond to a known function $F(x)$ defined on an open interval of the real line, which contains within it the open interval that goes from x_1 to x_N. In other situations, the numbers u_i are simply measured or observed values of a physical, economic, or other variable from which it is known that it can vary with x in a continuous manner, but without knowing the explicit function $F(x)$ that gives the value of

$$u = F(x)$$

through computation operations from the value of x. The problem in question is to estimate what the value is that corresponds to u in any point x contained in I, and that is not any of the x_i previously mentioned. It deals with, in other words, the interpolation of the variable between values $u_1, u_2, \dots, u_N$.

This problem has existed since the days of Newton, and he, as well as other greats of mathematical analysis such as Cotes, Vandermonde, Gauss, Lagrange, Hermite, and Tchebicheff dedicated extensive work to the subject from the 17th to the 20th centuries.

It was Newton who established the first great strategy to solve it, which consists in appealing to the class of whole polynomials in variable x, of degree *N-1* and searching within them for the polynomial (unique), that passes through points (x_1, u_1), (x_2, u_2), ... (x_N, u_N) on the Cartesian plane OXU. Newton himself has shown that this polynomial exists and is unique, only under the requirement (somewhat evident) that $x_1 \neq x_2 \neq \cdots \neq x_n$. We call it $p(x)$ as well as "the position polynomial" of data x_i with values u_i. Numerical Analysis books present several ways of writing this polynomial. One the polynomial $p_n(x)$, is known, it would be used (following Newton) to estimate the value of u on an arbitrary point on x. It is obvious that, given the definition of $p(x)$, the abscissae x_i and its corresponding values u_i meet the N identities.

$$\left.\begin{array}{l} p(x_1) = u_1 \\ p(x_2) = u_2 \\ \quad\vdots \\ p(x_N) = u_N \end{array}\right\} \qquad (1.01)$$

Vandermonde gave the position polynomial the very general and very elegant form that is expressed with the determinant that bears his name

$$\begin{vmatrix} p(x) & 1 & x & .... & x^{N-1} \\ y_1 & 1 & x_1 & .... & x_1^{N-1} \\ y_2 & 1 & x_2 & .... & x_2^{N-1} \\ \vdots & & & & \\ y_N & 1 & x_N & .... & x_N^{N-1} \end{vmatrix} = 0 \qquad (1.02)$$

But this determinant is cumbersome to handle in arithmetic and algebraic calculations. So, Lagrange found an equivalent, very general formula, (which bears his name) for the placement polynomial, and which is expressed in the form

$$p(x) = \sum_{i=1}^{i=N} \frac{(x-x_1)\,...\,(x-x_{i-1})\,(x-x_{i+1})\,...\,(x-x_N)}{(x_i-x_1)\,...\,(x_i-x_{i-1})\,(x_i-x_{i+1})\,...\,(x_i-x_N)}\, u_i \qquad (1.03)$$

This is Lagrange's famous formula for interpolation, which is more adaptable to numerical computation, and can be programmed without difficulty for a personal computer. It is very easy to prove that the right side of this formula (1.03) is a polynomial of degree *N-1* and that it satisfies the N conditions (1.01) written above. Therefore, it is the position polynomial of values u_i with values x_i, a polynomial that, as one knows, is unique although it can be presented in various forms.

The right-hand side of formula (1.03) is often abbreviated, coexisting in adopting the notation

$$\pi(x) = (x - x_1)(x - x_2)\,...\,(x - x_{N-1})\,(x - x_N) \prod_{i=1} \equiv (x - x_i) \qquad (1.04)$$

And furthermore

$$l_i(x) = \frac{(x-x_1)(x-x_i)\,...\,(x-x_{i-1})(x-x_{i+1})...(x-x_N)}{(x_i-x_1)\,(x_i-x_2)\,...\,(x_i-x_{i-1})\,(x_i-x_{i+1})...\,(x_i-x_N)} \qquad (1.05)$$

It becomes obvious that the denominator of $l_i(x)$ is

$$(x_i - x_1)(x_i - x_2)\,...\,(x_i - x_{i-1})(x_i - x_{i+1})\,...\,(x_i - x_n) = \pi'(x_i) \qquad (1.06)$$

$$= d\pi(x)/dx, \text{ valorado en } x = x_i \qquad (1.07)$$

And therefore $l_i(x)$ is the degree polynomial *N-1 given by*

$$l_i(x) \qquad \frac{\pi(x)}{(x - x_i)\,\pi'(x_i)} \qquad (1.08)$$

In this way, the Lagrange formula for the interpolating and position polynomials for the points (x_1, u_1), (x_2, u_2), ... (x_N, u_N) can be written

$$p(x) = \sum_{i=1}^{N} u_i \cdot l_i(x) \qquad (1.09)$$

which is the summarized form of **Equation 1.03**.

2. If it were known in advance that there is an explicit, analytically well-defined function that binds u and x in the form

$$u = F(x) \qquad (2.01)$$

which is numerically computable, it would be obvious that identities are fulfilled.

$$\left.\begin{array}{l} u_1 = F(x_1) \\ u_2 = F(x_2) \\ \quad\vdots \\ u_n = F(x_n) \end{array}\right\} \qquad (2.02)$$

and that on the N points $x_i (i = 1,2, \ldots, N)$, it would occur that the values of the function $F(x)$ coincide with those of the position polynomial $p(x_i)$:

$$\left.\begin{aligned} F(x_1) &= p(x_1) \\ F(x_2) &= p(x_2) \\ &\vdots \\ F(x_n) &= p(x_n) \end{aligned}\right\} \tag{2.03}$$

But, in general, at other points x of the real line, other than the network $x_1, x_2, \ldots x_N$ prescribed from the beginning, the function $F(x)$ and the polynomial $p(x)$ need not coincide. However, using Rolle's theorem and the Taylor series, in all numerical analysis books it is shown that for any x of the domain of $F(x)$, it would have equal value

$$F(x) = p(x) + \frac{1}{N!}\pi(x)F_{(z)}^{(N)} \tag{2.04}$$

where z is a point whose value cannot be determined, but which certainly exists and belongs to the open interval $I = (x_1 \ldots x_N)$. This can be seen in any of the books mentioned in the bibliography. In this way, it is possible to at least know how much the maximum limit is of the absolute error of $\varepsilon(x)$ that is committed when using $p(x)$ to estimate the value of u at a point x that is not of the points $x_1, x_2, \ldots x_N$. Indeed:

$$\max|\varepsilon(x)| = \frac{1}{N!} \max_{x\in I} \left|\pi(x)F_{(z)}^{(N)}\right. \tag{2.05}$$

as a brief reflection shows. Thus, if we know $F(x)$ and its derivatives to $F_{(x)}^{(N)}$, the Lagrange formula can be written:

$$u(x) = F(x) = \sum \frac{(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_N)}{(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_N)} u_i + \varepsilon(x) \tag{2.06}$$

and the "error" is bounded by the formula (2.05).

But if $F(x)$ and its derivatives are not known, the most that can be written is that

$$u(x) = p(x) + \varepsilon(x) \tag{2.07}$$

where $\varepsilon(x_1) = 0 = \varepsilon(x_2) = \cdots \varepsilon(x_N) = 0$, and those who use math for worldly purposes write

$$u(x) = \simeq p(x) \text{ en } x \in I \tag{2.08}$$

with the permission of formalist orthodoxy.

3. THE PLANE ON $\mathbb{R}^3$ THAT PASSES THROUGH THREE POINTS

3. In order to generalize Lagrange's formula for several variables, let's recall the following result from analytic geometry in three-dimensional space.

3. On $\mathbb{R}^3$ there are three distinct points $Q_1(x_1, y_1, z_1)$, $Q_2(x_2, y_2, z_2)$, $Q_3(x_3, y_3, z_3)$, and we need to establish the value of the coefficients of the plane

$$ax + by + cz + 1 = 0 \tag{3.01}$$

that passes through the three points mentioned. It is obvious, then, that such coefficients are the numbers that fulfill the condition

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \tag{3.02}$$

taught by analytic geometry. The projections of Q_1, Q_2, Q_3 on the plane OXY are called $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$.

Developing the determinant (3.02) through the Laplace rule, we find that

$$z = \frac{\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\Delta} z_1 - \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\Delta} z_2 + \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\Delta} z_3 \tag{3.03}$$

and in this expression the symbol Δ is the determinant

$$\Delta = \begin{vmatrix} x & y & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{3.04}$$

By executing algebraic operations and simplifying terms, it turns out that the plane searched for, which passes through Q_1, Q_2 and Q_3, is the plane

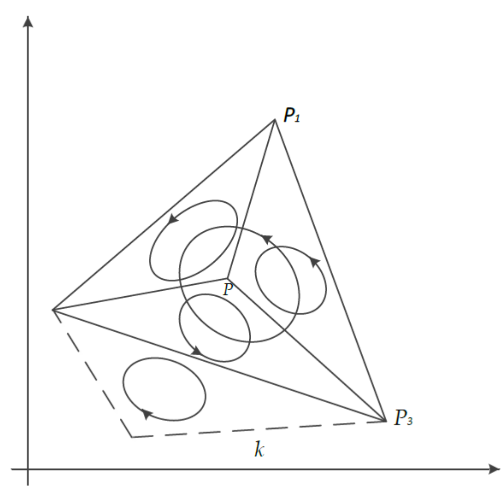
$$z(x,y) = \frac{(x-x_2)(y-y_3) - (x-x_3)(y-y_2)}{(x_1-x_2)(y_1-y_3)-(x_1-x_3)(y_1-y_2)}\, z_1 \; +$$

$$+ \; \frac{(x-x_3)(y-y_1) - (x-x_1)(y-y_3)}{(x_2-x_3)(y_2-y_1)-(x_2-x_1)(y_2-y_3)}\, z_2 \; + \qquad (3.05)$$

$$+ \; \frac{(x-x_1)(y-y_2) - (x-x_2)(y-y_1)}{(x_3-x_1)(y_3-y_2)-(x_3-x_2)(y_3-y_1)}\, z_3 = L(x,y)$$

Now it can be noted that both the numerators and denominators of the fractions on the right side of (2.05) have a clear geometrical meaning, which is shown in the attached figure.

Figure 1



In this figure, we have the plane OXY and the three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$. The generic point $P_n(x, y)$ is the projection in this plane of a generic point $Q(x, y, z)$, located in the plane to be searched.

The area of the triangle $P_1P_2P_3$ is, as is well known

$$A_{123} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \qquad (3.06)$$

and it is easy to demonstrate that if P_1, P_2, P_3 are in the left-hand order, the determinant of (2.06) is a positive number. This determinant can be developed and written in three ways:

$$\begin{aligned} A_{123} &= (1/2)[(x_1-x_2)(y_1-y_3)-(x_1-x_3)(y_1-y_2)] \\ &= (1/2)\,[(x_2-x_3)(y_2-y_1)-(x_2-x_1)(y_2-y_3)] \qquad (3.07) \\ &= (1/2)\,[(x_3-x_1)(y_3-y_2)-(x_3-x_2)(y_3-y_1)] \end{aligned}$$

Furthermore, the area of the triangle $P\,P_2\,P_3$ is

$$A_{023} = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad (3.06)$$

$$= (1/2)[(x-x_2)(y-y_3)-(x-x_3)(y-y_2)] \qquad (3.08\ \text{A})$$

We can see also that

$$(1/2)[(x-x_3)(y-y_1)-(x-x_1)(y-y_3)] = (1/2)\begin{vmatrix} x_1 & y_1 & 1 \\ x & y & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (1/2)\,A_{103} = (1/2)A_{103} \qquad (3.09)$$

where

A_{103} = Area (oriented, that is, signified) of the triangle $P_1P\,P_3$, traversing in that order of its vertices

Furthermore,

$$(1/2)[(x-x_1)(y-y_2)-(x-x_2)(y-y_1)] = (1/2)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = (1/2)\,A_{120} \qquad (3.10)$$

where

A_{120} = the oriented (and signified) area of the triangle $P_1P\,P_3$, traversed in that order of its vertices

Therefore, the equation of the plane passing through Q_1, Q_2, Q_3 can also be written as

$$z(x,y) = \frac{A_{023}}{A_{123}}\,z_1 + \frac{A_{103}}{A_{123}}\,z_2 + \frac{A_{120}}{A_{123}}\,z_3 = L(x,y) \qquad (3.11)$$

The signs of the areas of the various triangles in view are determined by a rule well known in three-dimensional analytic geometry, which is as follows.

The points Q_1, Q_2, Q_3 are listed so that their projections $P_1P_2P_3$, traversed in this order, determine a left-handed triangle whose area A_{123} we agree is positive.

Under these conditions, the three areas A_{023}, A_{103}, A_{120} have positive or negative signs, depending on whether their respective triangles $P\,P_2P_3$, P_1PP_3, P_1P_2P indicate, with the order of their three sub-indices, a left-handed path or a right-handed path.

By going to the determinants that define A_{023}, A_{103}, A_{120}, or to an elementary drawing, we find that, whatever the signs are in these three areas, we have to

$$A_{023} + A_{103} + A_{120} = A_{123} \tag{3.12}$$

or

$$A_{023}/A_{123} + A_{103}/A_{123} + A_{120}/A_{123} = 1 \tag{3.12.A}$$

The coefficients A_{023}/A_{123}, A_{103}/A_{123}, A_{120}/A_{123} can be:

a. All positive, when P is inside $P_1P_2P_3$.

b. Two positive and one negative. For example, in **Figure 2** it is

$$A_{023} > 0 \;;\; A_{103} > 0 \;;\; A_{120} < 0$$

c. One positive and two negatives. For example, in **Figure 3** it is

$$A_{023} < 0 \;;\; A_{103} > 0 \;;\; A_{120} < 0$$

Figure 2

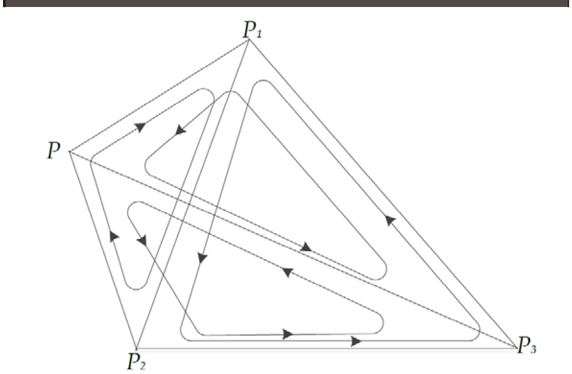
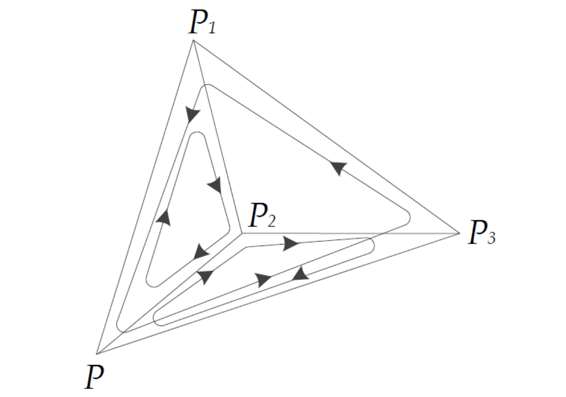


Figure 3



The numbers A_{023}/A_{123}, A_{103}/A_{123} y A_{120}/A_{123} are called, in geometry, the trilinear coordinates of the point P referring to the triangle $P_1P_2P_3$.

It is also very easy to prove or verify that if, in any of the three numbers A_{023}, A_{103}, A_{120} an even permutation is made of said number's subscripts, it would not vary in absolute value or sign.

$$\left.\begin{array}{l} A_{023} = A_{302} = A_{230} < 0 \quad ; \quad A_{103} = A_{310} = A_{031} > 0 \\ A_{120} = A_{012} = A_{201} < 0 \end{array}\right\} \tag{3.13}$$

But if an odd permutation is made, those numbers change sign:

$$\left.\begin{array}{l} A_{023} = -A_{203} = -A_{032} \quad ; \quad A_{103} = --A_{310} = A_{031} \\ A_{120} = -A_{012} = -A_{201} \end{array}\right\} \tag{3.14}$$

The signs of the areas can also be established according to the following rules:

a. The area A_{123} will be taken as positive (although the vertices $P_1P_2P_3$ are in a right-hand sequence).

b. Each area A_{oij} (where *i,j* are a pair taken from the list 1,2,3) has its sign like this:

- If $(P_oP_iP_j) \cap (P_1P_2P_3)$ is not empty, it will be put as $sign\ A_{oij} = +1$
- If $(P_oP_iP_j) \cap (P_1P_2P_3)$ is empty, it will be put as $sign\ A_{oij} = -1$

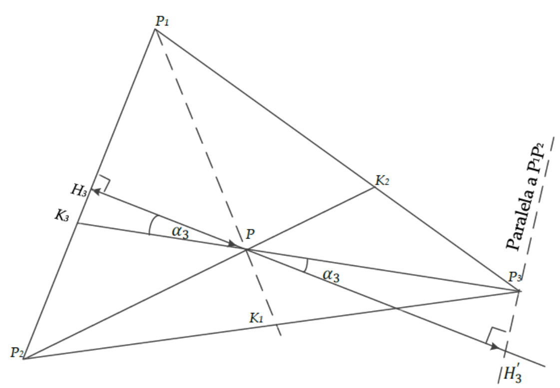
and this regardless of the order in which we write the three subscripts.

It should be noted that each area A_{ijk}

$$A_{ijk} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

retains its algebraic and numerical value, including the sign, when an even permutation of the subscripts is made, as well as when it makes a parallel translation of the coordinate system, and even also when this coordinate system rotates in the plane. These properties are shown algebraically, and are checked by mere geometric observation.

Figure 4



In **Figure 4** it can be seen immediately that, by the similarity of the triangles, it can be written that

$$\frac{A_{012}}{A_{312}} = \frac{0.5 \times P_1P_2 \times H_3P}{0.5 \times P_1P_2 \times H_3P_3} = \frac{\overline{H_3P}}{\overline{H_3P_3}} = \frac{\overline{k_3P}}{\overline{k_3P_3}}$$

and in a similar manner

$A_{023}/A_{123} = \overline{k_1P} \,/\, \overline{k_1P_1}$ = trilinear coordinate of P in direction z_1

$A_{031}/A_{123} = \overline{k_2P} \,/\, \overline{k_2P_2}$ = trilinear coordinate of P in direction z_2

4. In summary, we have the following theorem: for the three points

$$Q_1(x_1, y_1, z_1) \ , Q(x_2, y_2, z_2) \ , Q(x_3, y_3, z_3)$$

given in the Euclidean space $\mathbb{R}^3$, whose projections in the plane OXY are

$$P_1(x_1,y_1) \quad , \quad P_2(x_2,y_2) \quad , \quad P_3(x_3,y_3)$$

and whose coordinates do not nullify the determinant (3.04)

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

for these three points Q_1, Q_2, Q_3 passes a single plane of $\mathbb{R}^3$, whose equation is first degree in x and first degree in y, and which can be written in the form found in (3.10)

$$z(x,y) = \frac{A_{023}}{A_{123}} z_1 + \frac{A_{103}}{A_{123}} z_2 + \frac{A_{120}}{A_{123}} z_3 \qquad (4.01)$$

where:

$$\left.\begin{array}{ll} A_{123}: & \text{Area of the reference triangle } P_1P_2P_3\text{, which we consider positive} \\ P(x,y): & \text{Generic point of } OXY \text{ which is projection of the generic point } Q(x,y,z)\text{, which runs through the plane we describe} \\ A_{102}: & \text{Area of triangle } P_0P_1P_2\text{, with sign given by (3.13) and (3.14)} \\ A_{103}: & \text{Area of the triangle } P_0P_1P_2\text{, with sign given by (3.13) and (3.14)} \\ A_{120}: & \text{Area of the triangle } P_0P_1P_2\text{, with sign given by (3.13) and (3.14)} \\ A_{123} = A_{012} + A_{103} + A_{120} & \end{array}\right\} \quad (4.02)$$

Said equation can also be written

$$z(x,y) = \frac{\overline{k_1P}}{\overline{k_1P_1}} z_1 + \frac{\overline{k_2P}}{\overline{k_2P_2}} z_2 + \frac{\overline{k_3P}}{\overline{k_3P_3}} z_3 \qquad (4.03)$$

Where segments $\overline{k_1P}$, $\overline{k_2P}$, $\overline{k_3P}$ carry "plus" or "minus" signs according to conventions (3.13) and (3.14).

If in the equation

$$z(x,y)=\frac{A_{023}}{A_{123}}z_1+\frac{A_{103}}{A_{123}}z_2+\frac{A_{120}}{A_{123}}z_2 \qquad (4.01\text{ B})$$

the point $P(x, y)$ moves to match $P_1(x, y)$. It can be seen that **Figure 3** and Formula (3.03) say that when

$$P(x,y)\to P_1(x_1,y_1) \text{ then } \begin{aligned} A_{023} &\to A_{123}\\ A_{103} &= A_{031} \to 0\\ A_{120} &= A_{012} \to 0\end{aligned}$$

and that, consequently, when P coincides with $P_{1,}$ the **Equation 3.10.B** reduces to the identity

$$z(x, y) = z_1$$

as was required from the outset.

By analogous considerations we can verify that

$$z(x_2, y_2) = z_2 \qquad \text{y que} \qquad z(x_3, y_3) = z_3$$

It is well known that the three medians of the triangle $P_1P_2P_3$ are cut at the same point, located at 2/3 of their lengths, and measured from the respective vertices. That point coincides with the center of gravity G of the triangle, and with the vertices determines, three triangles which are

$$z(x_G, y_G) = (z_1 + z_2 + z_3)/3$$

as is only made just clear.

4. LAGRANGE'S FORMULA FOR FOUR POINTS ON THE PLANE

5. We now deal with the case of 4 distinct, arbitrary, fixed points in plane $\mathbb{R}^2$, with known Cartesian coordinates:

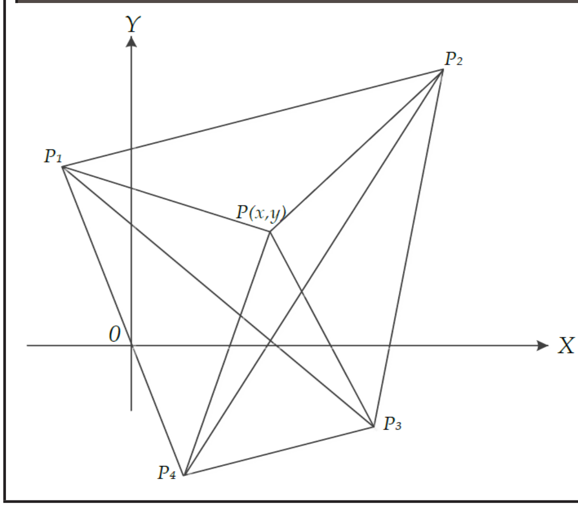
$$P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_2, y_3), P_4(x_4, y_4)$$

as shown in the attached figure. Also, in that region of the plane there is a continuous and differentiable function $f(x, y)$, which, in these points adopts the four values

$$f(x_1,y_1)=z_1 \;;\; f(x_2,y_2)=z_2 \;;\; f(x_3,y_3) = z_3 \;;\; f(x_4,y_4)$$

We then have a variable, arbitrary point on the same plane OXY, which we will call $P(x, y)$.

Figure 5



We consider all the triangles that can be formed with the 4 given points, and that have a vertex in P_1. The number of such triangles is the same as the combinations of 4-1 which are points, other than P_1, taken two by two (regardless of order), which is

$$\binom{4-1}{2} = 3$$

Let us construct the triangles that form with a vertex in P, in P_1 and the different pairs of the other vertices P_2, P_3, P_4. The areas of these triangles will be called, as above:

A_{132} = Area of the triangle $P_1P_2P_3$ (which we take with positive sign)

A_{142} = Area of the triangle $P_1P_2P_4$ (which we take with positive sign)

A_{1ij} = Area of the triangle $P_1P_iP_j$ (which we take with positive sign)

A_{032} = Area of the triangle PP_3P_2 (which has a positive sign)

and, in general,

A_{oij} = the area of the triangle PP_iP_j which is positive if the order of its vertices is left-handed and negative if that order is right-handed.

Guiding ourselves by the ideas of Lagrange, we construct the expression:

$$p(x,y) \equiv z_1 \frac{A_{023}}{A_{123}} \frac{A_{034}}{A_{134}} \frac{A_{024}}{A_{124}} + z_2 \frac{A_{034}}{A_{234}} \frac{A_{041}}{A_{241}} \frac{A_{031}}{A_{231}} + z_3 \frac{A_{041}}{A_{341}} \frac{A_{012}}{A_{312}} \frac{A_{024}}{A_{342}} + z_4 \frac{A_{012}}{A_{412}} \frac{A_{023}}{A_{423}} \frac{A_{013}}{A_{413}} \qquad (5.01)$$

which we can also write in a summarized manner as,

$$p(x,y) \equiv \sum_{i=1}^{i=N} z_1 \prod_{(j,k)}^{3} \frac{A_{ojk}}{A_{ijk}} \qquad (5.02)$$

where (i, j, k) is one of the three sets of three that can be extracted from the quatern (1,2,3,4). For the latter, and by definition, this must be $i \neq j$, $j \neq k$, $k \neq i$. The number of factors to the right of the output (∏) is $\binom{4-1}{2}$ =3. Formula (5.01), which we have summarized in (5.02), will be called the Lagrange interpolation formula for four points in the two dimensions OX, OY.

Furthermore, as we have already said:

$$A_{ijk} = \frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}; \quad A_{ojk} = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} \qquad (5.03)$$

These expressions show that numbers A_{ijk} are parameters that are determined by the four points P_1, P_2, P_3, P_4. Of these parameters, there are

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4$$

They are written in the denominators of the expression in (4.01).

On the other hand, for each fixed value of index i, the number of factors A_{ojk} is

$$\binom{4-1}{2} = \frac{3!}{2!\,1!} = 3$$

and this is how, effectively, each z_i is accompanied by three factors A_{ojk}. multiplying it. Furthermore, the equations in (5.03) show that each factor A_{ojk} is a trinomial in x, y, bilinear and non-homogenous, which means it has the form

$$A_{ojk} = a_{jk}\, x + b_{jk}\, y + c_{jk}$$

Therefore, the product of three of them is a polynomial of the third degree in x, y, that is, it takes the form

$$\begin{aligned} p(x,y) = &\alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3 + \\ &+ \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{o2}y^2 + \alpha_{10}x + \alpha_{o1}y + \alpha_{oo} \end{aligned} \qquad (5.04)$$

Hence the expression in (5.01) is a third-degree polynomial in x and third degree in y, which can have up to 10 numerical coefficients.

To calculate what values the expression takes at the four given reference points (P_1, P_2, P_3, P_4), note that, when P tends to P_k, or P_j the parameter A_{ojk} tends to zero, because

$$A_{jjk} = 0 \qquad A_{kjk} \qquad (5.05)$$

Therefore, when we set $P(x, y)$ to match $P_1(x_1, y_1)$, the polynomial in (5.01) takes the form

$$p(x_1, y_1) = z_1 \qquad (5.06.1)$$

The same considerations lead us to find that

$$p(x_2, y_2) = z_2 \qquad (5.06.2)$$

$$p(x_3, y_3) = z_3 \qquad (5.06.3)$$

$$p(x_4, y_4) = z_4 \qquad (5.06.4)$$

This means that the polynomial (5.04), which contains up to ten coefficients, passes through the four points $Q_1(x_1, y_1, z_1)$, $Q_2(x_1, y_1, z_1)$, $Q_3(x_3, y_3, z_3)$, $Q_4(x_4, y_4, z_4)$, located in the three-dimensional space $\mathbb{R}^3$. The projections of these points are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$.

Thus, $p(x, y)$ satisfies the given 4 points. But we can no longer say that it is the third-degree polynomial that passes through such points.

6. It should be remembered that the polynomial in x, y, of the four points Q_1, Q_2, Q_3, Q_4, which is also isotropic in both variables, is

$$f_{11}(x, y) = \beta_{11}xy + \beta_{10}x + \beta_{01}y + \beta_{00} \qquad (6.01)$$

Since its four coefficients β_{ij} are univocally defined by the four conditions

$$\left.\begin{aligned} \beta_{11}x_1y_1 + \beta_{10}x_1 + \beta_{01}y_1 + \beta_{00} &= z_1 \\ \beta_{11}x_2y_2 + \beta_{10}x_2 + \beta_{01}y_2 + \beta_{00} &= z_2 \\ \beta_{11}x_3y_3 + \beta_{10}x_3 + \beta_{01}y_3 + \beta_{00} &= z_3 \\ \beta_{11}x_4y_4 + \beta_{10}x_4 + \beta_{01}y_4 + \beta_{00} &= z_4 \end{aligned}\right\} \qquad (6.02)$$

It is also known that the polynomial (5.01) is defined, implicitly but univocally by equation

$$\begin{vmatrix} f_{11}(x, y) & xy & x & y & 1 \\ z_1 & x_1y_1 & x_1 & y_1 & 1 \\ z_2 & x_2y_2 & x_2 & y_2 & 1 \\ z_3 & x_3y_3 & x_3 & y_3 & 1 \\ z_4 & x_4y_4 & x_4 & y_4 & 1 \end{vmatrix} = 0 \qquad (6.03)$$

whose determinant is often called the Vandermonde determinant.

There is also a unique placement polynomial (with four coefficients) for the four points Q_1, Q_2, Q_3, Q_4 that has the quadratic form in x and linear in y.

$$f_{20}(x, y) = \gamma_{20}x^2 + \gamma_{10}x + \gamma_{01}y + \gamma_{00} \qquad (6.04)$$

but which is anisotropic.

This is determined univocally by four equations analogous to those in (5.02) or by a Vandermonde equation analogous to that in (6.03), except that the column in x is changed by a column in x^2.

There is another polynomial

$$f_{02}(x, y) = \delta_{02}y^2 + \delta_{01}y + \delta_{10}x + \delta_{00} \qquad (6.05)$$

which is linear in x and quadratic in y, anisotropic, which is univocally determined by four other conditions of the form of the equations in (5.02), and given implicitly and univocally by another Vandermonde equation similar to (5.03), but where column in xy is changed by a column in y^2.

7. Turning now to the function $f(x, y)$ presented at the beginning of number 5, it is now clear that we can write it in the form

$$f(x, y) = p(x, y) + \emptyset(x) \prod A_{ohk} \qquad (7.01)$$

Where (h, k) is one of the six pairs that can be extracted from the quatern (1,2,3,4). In the above expression, we are writing the interpolation error $\varepsilon(x, y)$ as

$$\varepsilon(x, y) = \emptyset(x) \cdot A_{023}(x, y) \cdot A_{034}(x, y) \cdot A_{024}(x, y) \cdot A_{041}(x, y) \cdot A_{031}(x, y) \cdot A_{012}(x, y)$$

because the latter term fulfills the conditions required for

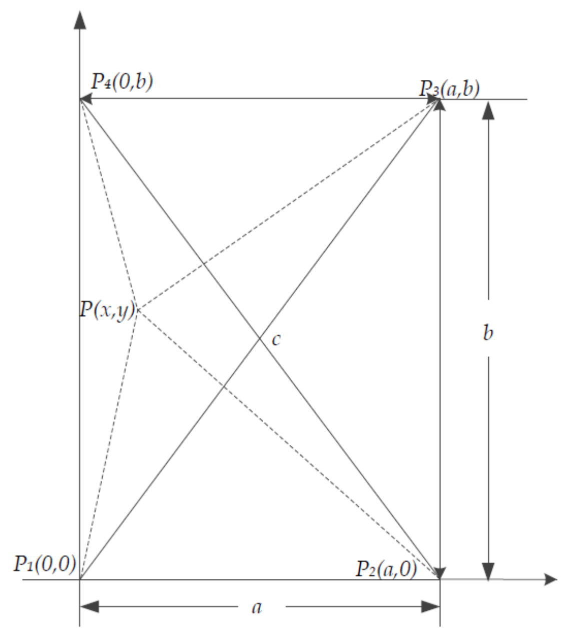
$$f(x_1, y_1) = p(x_1, y_1) = z_1 \;\; ; \;\; f(x_2, y_2) = p(x_2, y_2) = z_2$$
$$f(x_3, p_3 = p(x_3, y_3) = z_3 \;\; ; \;\; f(x_4, y_4) = p(x_4, y_4) = z_4$$

8. Example. Consider, as an example of the above, a function $f(x, y)$, defined in a region of the plane containing the four points

$$P_1(0,0) \;\; , \;\; P_2(a,0) \;\; , \;\; P_3(a, b) \;\; , \;\; P_4(0,b)$$

shown in the attached figure.

Figure 6



Consider also a point $P(x, y)$ contained in that same region; and construct the Lagrange formula to interpolate a function $f(x, y)$ whose values at the four points P_1, P_2, P_3, P_4 are respectively

$$f(0,0)=z \;,\; f(a,0)=z_2 \;,\; f(a,b)=z_3 \;,\; f(0,b)=z_4 \tag{8.01}$$

We make the necessary triangulation for formula (5.01), as seen in the neighboring figure. And we calculate the areas of these triangles, which are almost evident

$$\begin{gathered} A_{123}=ab/2 \quad A_{134}=ab/2 \quad A_{124}=ab/2 \\ A_{023}=b\,(a-x)/2 \quad A_{034}=a\,(b-y)/2 \\ A_{024}=b[a(b-y)/b-x]=A_{042} \\ A_{041}=bx/2 \;\; A_{013}=-A_{031}=b(ay/b-x) \\ A_{012}=ay/2 \end{gathered} \tag{8.02}$$

But, as already seen:

$$\begin{aligned} p(x,y) &= z_1\frac{A_{023}}{A_{123}}\frac{A_{034}}{A_{134}}\frac{A_{024}}{A_{124}}+z_2\frac{A_{034}}{A_{234}}\frac{A_{041}}{A_{241}}\frac{A_{031}}{A_{231}}+ \\ &\quad +z_3\frac{A_{041}}{A_{341}}\frac{A_{012}}{A_{312}}\frac{A_{042}}{A_{342}}+z_4\frac{A_{012}}{A_{412}}\frac{A_{023}}{A_{423}}\frac{A_{013}}{A_{413}} \end{aligned} \tag{8.03}$$

Substituting the values of the areas (with their corresponding "plus" or "minus" sign), you have:

$$\begin{aligned} p(x,y) &= z_1\frac{b(a-x)}{ab}\frac{a(b-y)}{ab}\frac{b[a(b-y)/b-x]}{ab} \\ &\quad + z_2\frac{a(b-y)}{ab}\frac{bx}{ab}(-1)\frac{b(ay/b-x)}{ab} \\ &\quad + z_3\frac{bx}{ab}\frac{ay}{ab}(-1)\frac{b[a(b-y)/b-x]}{ab}+z_4\frac{ay}{ab} \\ &\quad \frac{b(a-x)}{ab}\;\frac{b(ay/b-x)}{ab} \end{aligned} \tag{8.04}$$

Performing the operations indicated, simplifying numerators with denominators, and reducing similar terms, one finds:

$$\begin{aligned} p(x,y) &= (1/a^2b)\,[x^2\;y\,(-z_1-z_2+z_3+z_4)- \\ &\quad (a/b)\;xy^2(z_1-z_2-z_3+z_4)+ \\ &\quad +bx^2(z_1+z_2)+axy(3z_1-z_2-z_3-z_4)+ \\ &\quad (a^2/b)\,y^2\;(z_1+z_4)- \\ &\quad -2abxz_1\;-2a^2yz_1\;+a^2bz_1] \end{aligned} \tag{8.05}$$

This polynomial has eight terms, with their respective coefficients, in

$$x^2y,\, xy^2,\, x^2,\, xy,\, y^2,\, x,\, y,\, x^0y^0$$

It is, therefore, an isotropic polynomial.

The formula in (5.01) allows us to write:

$$p(x,b)=z_3(x/a)^2+z_4(1-x/a)^2\;, \qquad \text{on side } P_3P_4$$

$$p(0,y)=z_1(1-y/b)^2+z_4(y/b)^2\;, \qquad \text{on side } P_4P_1$$

$$p(x,bx/a)=z_1(1-a/x)^2\,(1-2x/a)+z_3x^2(2x/a-1)\;, \qquad \text{on the diagonal } P_1P_2$$

In the center of the rectangle is $x=a/2$, $y\;b/2$ and thus $p(a/2,b/2)=0$

in the center of the rectangle the Lagrange interpolator polynomial cancels out.

9. Another formula for the rectangle P_1, P_2, P_3, P_4. An interpolation formula unequivocally determined by the four vertices of any quadrilateral, it must have four coefficients. This could be in the form

$$z = c_{00} + c_{10} \cdot x + c_{01} \cdot y + c_{11} \cdot xy$$

if we ask that the interpolation be isotropic, that is, that it does not attribute powers to one of the two variables that the other does not have, in the formula. Thus, as already explained, we have the Vandermonde determinant.

$$\begin{vmatrix} z(x,y) & 1 & x & y & xy \\ z_1 & 1 & x_1 & y_1 & x_1y_1 \\ z_2 & 1 & x_2 & y_2 & x_2y_2 \\ z_3 & 1 & x_3 & y_3 & x_3y_3 \\ z_4 & 1 & x_4 & y_4 & x_4y_4 \end{vmatrix} = 0$$

It is easily calculated that $p(x, y)$ reproduces the values of $f(x, y)$ in the four corners of the rectangle of the figure. In fact, by substituting the corresponding coordinates in formula (8.04) or its equivalent form (8.05), and doing the indicated algebraic operations, we find that

$$p(0,0) = z_1 = f(0,0) \quad p(a,0) = z_2 = f(a,0)$$
$$p(a,b) = z_3 = f(a,b) \quad p(0,b) = z_4 = f(0,b)$$

On the points of the perimeter and on the diagonals, we have:

$a)$ on P_1P_2: $y=0$; $b)$ on P_2P_3: $x=a$;
$c)$ on P_3P_4: $y=b$; $d)$ on P_4P_1: $x=0$
$e)$ on P_1P_3; $y=bx/a$ $f)$ on P_2P_4 : $y = b(1 - x/a)$

Therefore, the interpolation formula (8.04) adopts the following expressions:

$p(a,0) = z_1(1 - x/a)^2 + z_2(x/a)^2$, on side P_1P_2

$p(a,y) = z_2(1 - y/b)^2 + z_3(y/b)^2$, on side P_2P_3

5. THE LAGRANGE FORMULA IN VARIOUS DIMENSIONS

In the study of many issues and problems of geography, meteorology, economics, demography, and other real-world sciences, the situation arises in which several data (in general, m data) are known for each of N points, points in which a variable , which depends on those data, takes known values; and it is a question of determining the value that has in another point—different from those already known—from their values at known points. For example, it would be the case of a geographer who studies the distribution of ambient temperatures in a country, and has the value of that temperature in ten places $(N = 10)$, and for each place he knows five data $(m = 5)$ on which its temperature depends: latitude, height above sea level, relative humidity, rainfall and sunshine.

The geographer's problem would be to determine by numerical calculation the average annual temperature in another location of that country, different from the cities mentioned.

In this paper, we present a method to construct a version of the Lagrange formula in a space of several dimensions (in general, of dimensions) and to use it in numerical calculation with real variables.

To say it formally: We have points in a space $\mathbb{R}^m$ formed by m real variables x^1, x^2, ...,x^m. Such points are $P_1(x_1^{\,1}, x_1^{\,2}, \dots, x_1^{\,m})$, ... $P_i\,(x_i^{\,1}, x_i^{\,2}, \dots, x_i^{\,m})$, ... , $P_N(x_N^{\,1}, x_N^{\,2}, \dots, x_N^{\,m})$ and belong to a space that is endowed with the usual Euclidean metric. At those points, and in a continuous and compact region D containing them, a function is defined

$$u(x^1, x^2, \dots, x^m)$$

which is continuous and differentiable, and which adopts N (known) numerical values.

$$u_i = u(x_i^1, x_i^2, \dots, x_i^m) \text{ para } i \epsilon \{1,2,\dots,N\}$$

In the space $\mathbb{R}^m$ a "topological simplex" is the name of a hyper-triangle (or hyperpolyhedral) which has m dimensions, is convex, and which has $m + 1$ vertices and $m + 1$ hyper-faces (hyper-planes). For example: in the usual three-dimensional Euclidean space, it is $m = 3$ and, a simplex in it, it is a tetrahedron (regular or not).

With the N points $P_1, P_2, \dots, P_N$ as vertices, it is possible to construct a number (cardinal) of simplices

$$\binom{N}{m+1} = N!/(m+1)!\,(N-m-1)! = \mu(N,\ m)$$

Supposing, of course, that $N > m$.

In addition, given one of the points P_i (any but fixed), it is possible to construct from it a number of simplices

$$\binom{N-1}{m} = (N-1)!/m! \quad (N-1-m)! = v(N,m)$$

using the $N - 1$ remaining points as vertices of these simplices.

Note: Dividing μ/v results $\mu/v = N/(m+1)$; in ; and since $N \geq m+1$, the result in turn is that $\mu \geq v$.

Each of these latter simplices is identified by its vertices in the form

$$P_1 P_{ji1}\ P_{ji2}\ \dots P_{jim} \text{ being each } jih \neq i$$

and its Euclidean hyper-volume will be represented as

$$V(P_i P_{ji1}\ \dots P_{jim})$$

where $ji1, ji2, \dots, jim$ is an ordered permutation extracted from the ordered sequence

$$\{1,2,\dots,i-1,i+1,\dots,N\}$$

in which the subscript "i" of P_i has been deleted.

Multidimensional Analytic geometry teaches that

$$V(P_i P_{ji1} \dots P_{jim}) = \frac{1}{m!} \begin{vmatrix} x_i^1 & x_i^2 & \dots. & x_i^m & 1 \\ x_{ji1}^1 & x_{ji1}^2 & \dots. & x_{ji1}^m & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{jim}^1 & x_{jim}^2 & \dots. & x_{jim}^m & 1 \end{vmatrix}$$

whose determinant is order $(m+1) \times (m+1)$. In each particular case, this determinant can be positive or negative, but it will never be null. That is: $V(P_j P_{ji1} \dots P_{jim}) \gtreqless 0$, but it always holds that $V \neq 0$.

To interpolate the variable in a point

$$P(x^1, x^2, \dots, x^m)$$

of $\mathbb{R}^m$, other than the points P_i the function $u(x^1, x^2, \dots x^m)$ can be estimated using the placement polynomial

$$u(x^1, \dots x^m) \simeq \sum \prod \frac{V(P\,P_{ji1} \dots P_{jim})}{V(P_i P_{ji1} \dots P_{jim})} u_i = p(x^1, x^2, \dots, x^m)$$

This is the general formula for interpolating the function sought. It can be called the Lagrange formula in m dimensions (or in $\mathbb{R}^m$). The polynomial $p(x', \dots x^m)$ is an algebraic, nonhomogeneous function of degree $N - 1$ (or less, in certain exceptional cases).

Strictly speaking, the above formula should be written as

$$u(x^1, \dots, x^m) = p(x', \dots x^m) + \epsilon(x', \dots, x^m)$$

where $\epsilon(x', \dots, x^m)$ is the interpolation error it is of the form:

$$\epsilon(x', \dots, x^m) = \prod_{i=1}^{N} \prod_{ji_m, \dots j_{im}}^{v} [V(P_i P_{ji1} \dots P_{jim}) - V(P\,P_{ji1} \dots P_{jim}] \emptyset(P)/N!$$

an expression that quickly tends towards zero when N increases, that is, when more and more points are taken as the basis for doing the interpolation.

6. AN ELEMENTARY EXAMPLE IN $\mathbb{R}^2$

We have the four points known in the plane $\mathbb{R}^2$, given by the coordinates

$$P_1(0{,}0) \qquad P_2(a, 0)$$

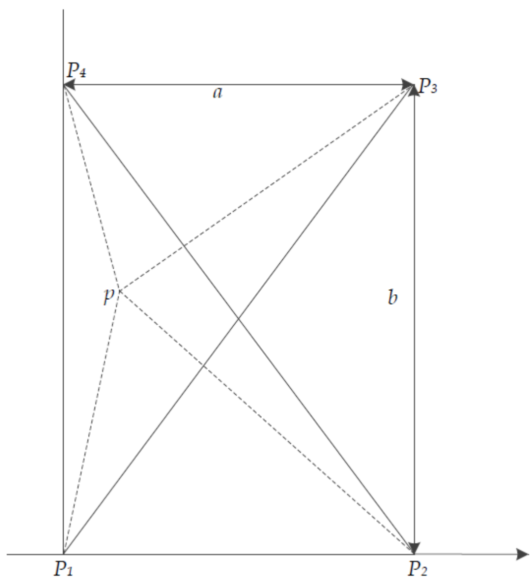
$$P_3(a, b) \qquad P_4(0, b)$$

And it is known that a continuous variable across the plane (e.g. temperature, topography, population density, etc.) takes the known values at those points

$$z_1 \qquad z_2$$

$$z_3 \qquad z_4$$

Figure 7



This refers to valuing z at the point

$$P(x, y)$$

In this case ($\mathbb{R}^2$, $m = 2$) the simplices that are necessary and sufficient to calculate z in P, are four triangles

$$P_1P_2P_3 \quad , \quad P_1P_2P_4 \quad , \quad P_2P_3P_4 \quad , \quad P_3P_4P_1$$

whose areas measure:

$$A_{123}=ab/2 \ , \ A_{234}=ab/2 \ , \ A_{341}=ab/2 \ , \ A_{412}=ab/2$$

(These areas correspond to what in the previous section were designated as "hypervolumes" and the four triangles are simplices in the plane).

From the point $P(x, y)$ we can form four triangles (which are simplices in $\mathbb{R}^2$) with the polygon $P_1P_2P_3P_4$ Their areas are::

$$A_{012} = a \cdot y/2 \qquad A_{023} = b(a-x)/2$$

$$A_{034} = a(b-y)2 \qquad A_{041} = b\,x)/2$$

And so, the placement polynomial is

$$p(x, y) = \sum_{i=1}^{i=4} \prod_{\substack{j\neq i \\ k\neq i}} \frac{A_{0jk}}{ijk} =$$

$$= z_1 \frac{A_{023}}{A_{123}} \frac{A_{034}}{A_{134}} + z_2 \frac{A_{034}}{A_{234}} \frac{A_{041}}{A_{241}} + z_3 \frac{A_{041}}{A_{341}} \frac{A_{012}}{A_{312}} + z_4 \frac{A_{012}}{A_{412}} \frac{A_{023}}{A_{423}}$$

And the formula for interpolating values of z in points of the rectangle in the drawing, is:

$$z(x,y) \simeq p(x, y) = \frac{1}{ab}[z_1(b-y)(a-x)+z_2x(b-y)+z_3xy+z_4(a-x)y]$$

that is to say, a convex combination of the four values z_1, z_2, z_3, z_4.

7. CALCULATION ALGORITHM

The procedure for calculating interpolated values in a region of a real space of several dimensions ($\mathbb{R}^m$) can follow the following algorithm.

The problem is considered in a space of m dimensions, where

a. N points $P_1, P_2, \ldots, P_N$, of known coordinates.

b. The N numerical values that have a variable u in those points, and which are: $u_1, \ldots, u_N$ respectively.

And it is a matter of calculating the value that the variable u en un punto P adopts (approximate, or exact) at a point N other than the given points.

Procedure:

1. Enter the $N \times m$ numerical coordinates of the points P_i:

$x^1, x_1{}^2, \dots x_1{}^m; x^1{}_2, x_2{}^2, \dots ; x_2{}^m; \dots . ; x_N{}^1 , x_N{}^2 , \dots , x_N{}^m$

of the N points. They are known data.

2. Enter the N values (also known)

$$u_1, u_2, \dots , u_N$$

3. Take P_1 and form the $\binom{N-1}{m} = v$, body simplices of dimension m, which can be constructed with the remaining $N - 1$ points. They will be called $(P_1 P_{j11} P_{12} \dots P_{jm1}), \dots , (P_1 P_{j11} P_{12}, \dots , P_{j1m})$, where the subscripts $j_{1_K} \neq 1$ and each one adopts the values in the collection $\{1,2, \dots , N\}$ omitting the value 1.

4. Calculate the respective hypervolumes using the determinant that was presented above.

5. Form and numerically calculate the v quotients

$$V(PP_{j11} \dots P_{j1m}) \div V(P_1 P_{j11} \dots P_{j1m})$$

and multiply them together. The numerical result is q_1.

6. Take $P_2, P_3, \quad \dots , P_N$ successively and orderly, and do each one the three steps, 3, 4 and 5. The respective results are $q_2, q_3, \dots , q_N$.

7. Construct and calculate the sum numerically

$$q_1 u_1 + q_2 u_2 + \cdots + q_N u_N$$

8. The interpolated value sought is

$$u(P) \cong q_1 u_1 + \cdots + q_N u_N$$

8. CONCLUSIONS

- The problem of interpolating numerical values in tabular functions in regions of two or more variables remains a relevant problem, despite the current availability of manual calculators and high-speed computers.

- In the common literature on numerical analysis there are no algorithms that can be used for this purpose, unlike those that occur with functions of a single variable, for which there are numerous interpolation formulas, such as the Lagrange formula utilized in this document.

- The Lagrange formula in 1 dimension is used to calculate or estimate values $p(x)$ of a function of 1 independent variable whose numerical values are known in several points of x, which are $u(x_1), u(x_2), \dots , (x_N)$. The formula expresses $f(x)$ other than those known, such as x other than those known, such as

$$p(x) = \sum_{i=1}^{N} \frac{(x-x_1) \dots (x-x_{i-1})\,(x-x_{i+1}) \dots (x-x_N)}{(x_i-x_1) \dots (x_i-x_{i-N})\,(x_i-x_{i+1}) \dots (x_i-x_N)}\, u(x_i)$$

where the values of $u(x_i)$ are obtained by an explicit computable formula, or are observed empirical measurements.

- In the case of plane $\mathbb{R}^2$, if we have the values z_1, z_2, z_3 and z_4 of a dependent variable (or values observed empirically) at points P_1, P_2, P_3, P_4, a function of that same variable at another point $z(x, y)$ different from the ones known, with the formula deduced here, can be estimated by

$$z(x, y) = z_1 \frac{A_{023} \cdot A_{034}}{A_{123} \cdot A_{134}} + z_2 \frac{A_{034} \cdot A_{041}}{A_{234} \cdot A_{241}} + z_3 \frac{A_{041} \cdot A_{012}}{A_{341} \cdot A_{312}} + z_4 \frac{A_{012} \cdot A_{023}}{A_{412} \cdot A_{423}}$$

where each A_{ijk} is the positively oriented area of the triangle $P_i P_j P_k$ which is given by the determinant.

$$A_{ijk} = \frac{1}{2} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

- In the case of N points $P_1, P_2, \dots . , P_N$ of known coordinates, in a space $\mathbb{R}^m$ of m dimensions, encompassed by a continuous variable, u, dependent on m variables and whose numerical values

$$u_1 = u(P_1)\,, u_2 = u(P_2), \dots , u_N = u(P_N)$$

the value of u can be calculated or estimated at its point p neighboring those known by the formula

$$u(P) = q_1 \cdot u_1 + q_2 \cdot u_2 + \cdots + q_N u_N$$

where

$$q_h = \prod_{k=1}^{v} \frac{V(0, j_1, \dots, j_m)}{V(h, j_1, \dots, j_m)}$$

and $V(0, j_1, \dots, j_m)$ is the volume of a simplex (of dimension m) formed by the point P_h where it is being estimated and a combination of m points taken from the known N points. There are $\binom{N}{m} = v$ of these combinations. Each simplex is of m dimension and has $m + 1$ vertices and $m + 1$ faces.

- The volume of each simplex V_k mentioned, formed by the points P_k, P_{j1}, ... , P_{jm} and being x_{i1}, x_{i2}, ... , x_{im} the coordinates (or components) of each point P_i , is given by the determinant

$$\frac{1}{m!} \begin{vmatrix} x_{k1} & x_{k2} & \cdots & x_{km} & 1 \\ x_{11} & x_{12} & \cdots & x_{1m} & 1 \\ \vdots & \vdots & & & \\ x_{m1} & x_{m2} & \cdots & x_{mm} & 1 \end{vmatrix}$$

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